

# Lagrangian stable slow dynamos in compact 3D Riemannian manifolds

by

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## Abstract

Modifications on a recently introduced fast dynamo operator by Chiconne et al [Comm Math Phys 173, 379 (1995)] in compact 3D Riemannian manifolds allows us to shown that slow dynamos are Lagrangian stable, in the sense that the sectional curvature of the Riemann manifold vanishes. The stability of the holonomic filament in this manifold will depend upon the sign of the second derivative of the pressure along the filament and in the non-holonomic case, to the normal pressure of the filament. Lagrangian instability is also investigated in this case and again an dynamo operator can be defined in this case. Negative curvature (Anosov flows) dynamos are also discussed in their stability aspects. **PACS numbers:**

02.40

# I Introduction

In chaotic dynamos [1, 2], Lyapunov exponents [3] or Arnold [2] Zeldovich [4] stretching are strongly responsible for the dynamo instability in an infinitely conducting fluid flow, as discussed earlier by Friedlander and Vishik [5]. This fluid possesses a magnetic Reynolds number  $R_m \rightarrow \infty$  which is inversely proportional to the resistivity number  $\eta$ . Therefore to better understand the process of instabilities is important to investigate the relation between these dynamo operators and the instabilities in the dynamo flows. Since, as pointed out by Kambe [6] the Lagrangean instability of flows is one of the most important geometrical and topological instabilities in fluid dynamics, a thoroughly understanding of this relationship is certain important for dynamo theory. With this motivation, in this paper we define a new dynamo operator in 3D compact Riemannian manifold which as is shown in the form of the theorem 1 below, which help us to show that slow dynamos are stable in the Lagrangean sense [6]. Chiconne et al [7] have recently obtained a similar operator and investigated its fast dynamo spectrum and the corresponding spectrum of group acting in the space of continuous divergence free vector field  $\vec{u}$  and  $\vec{H}$  representing, respectively the flow and the magnetic field in this compact Riemann manifold. Here the Lagrangean stability in this very same space is computed and it is shown that the spectrum eigenvalue equation in the form of the self-induction equation, leads to stable slow dynamos. Slow dynamos are defined with constraint ( $p_0 = 0$ ), where the magnetic field is given by the exponential stretching  $\vec{H} = \vec{H}_0 e^{p_0 t}$  is given. As a corollary it is shown that in the case of magnetic twisted filaments the stability of the dynamos depend upon the second derivatives of the flow pressure along the filaments. From the physical viewpoint there are working examples of slow dynamos which are much easier to obtain than the kinematic fast dynamos. This discussion was first given by Soward and Childress [8] and Moffatt and Proctor [9]. The paper is organized as follows: In section 2 a review of Riemann sectional curvature is given. In section 3 dynamo operator spectrum is computed. In this same section magnetic filament examples in holonomic and nonholonomic cases are examined concerning its Lagrangean stability properties. Section 4 we present the conclusions.

## II Sectional Riemann curvature

In this section we make a brief review of the differential geometry of surfaces in coordinate-free language of differential geometry. The Riemann curvature is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \quad (\text{II.1})$$

where  $X \in T\mathcal{M}$  is the vector representation which is defined on the tangent space  $T\mathcal{M}$  to the 3D Riemannian manifold  $\mathcal{M}$ . Here  $\nabla_X Y$  represents the covariant derivative given by

$$\nabla_X Y = (X \cdot \nabla) Y \quad (\text{II.2})$$

which for the physicists is intuitive, since we are saying that we are performing derivative along the  $X$  direction. The expression  $[X, Y]$  represents the commutator, which on a vector basis frame  $\vec{e}_i$  in this tangent sub-manifold defined by

$$X = X_k \vec{e}_k \quad (\text{II.3})$$

or in the dual basis  $\partial_k$

$$X = X^k \partial_k \quad (\text{II.4})$$

can be expressed as

$$[X, Y] = (X, Y)^k \partial_k \quad (\text{II.5})$$

In this same coordinate basis now we are able to write the curvature expression (II.1) as

$$R(X, Y)Z := [R^l_{j k p} Z^j X^k Y^p] \partial_l \quad (\text{II.6})$$

where the Einstein summation convention of tensor calculus is used. The expression  $R(X, Y)Y$  which we shall compute bellow is called Ricci curvature. The sectional curvature which is very useful in future computations is defined by

$$K(X, Y) := \frac{\langle R(X, Y)Y, X \rangle}{S(X, Y)} \quad (\text{II.7})$$

where  $S(X, Y)$  is defined by

$$S(X, Y) := \|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2 \quad (\text{II.8})$$

where the symbol  $\langle, \rangle$  implies internal product.

### III Dynamo operator for twisted filaments

Let us now start by considering the MHD field equations and the framework for non-holonomic filaments [10]

$$\nabla \cdot \vec{H} = 0 \quad (\text{III.9})$$

$$\nabla \times \vec{H} = \partial_t \vec{H} \quad (\text{III.10})$$

the field  $\vec{H}$  along the filament. The vectors  $\vec{t}$  and  $\vec{n}$  along with binormal vector  $\vec{b}$  together form the Frenet frame which obeys the Frenet-Serret equations

$$\vec{t}' = \kappa \vec{n} \quad (\text{III.11})$$

$$\vec{n}' = -\kappa \vec{t} + \tau \vec{b} \quad (\text{III.12})$$

$$\vec{b}' = -\tau \vec{n} \quad (\text{III.13})$$

the dash represents the ordinary derivation with respect to coordinate  $s$ , and  $\kappa(s, t)$  is the curvature of the curve where  $\kappa = R^{-1}$ . Here  $\tau$  represents the Frenet torsion. We follow the assumption that the Frenet frame may depend on other degrees of freedom such as that the gradient operator becomes

$$\nabla = \vec{t} \frac{\partial}{\partial s} + \vec{n} \frac{\partial}{\partial n} + \vec{b} \frac{\partial}{\partial b} \quad (\text{III.14})$$

The other equations for the other legs of the Frenet frame are

$$\frac{\partial}{\partial n} \vec{t} = \theta_{ns} \vec{n} + [\Omega_b + \tau] \vec{b} \quad (\text{III.15})$$

$$\frac{\partial}{\partial n} \vec{n} = -\theta_{ns} \vec{t} - (\text{div} \vec{b}) \vec{b} \quad (\text{III.16})$$

$$\frac{\partial}{\partial n} \vec{b} = -[\Omega_b + \tau] \vec{t} - (\text{div} \vec{b}) \vec{n} \quad (\text{III.17})$$

$$\frac{\partial}{\partial b} \vec{t} = \theta_{bs} \vec{b} - [\Omega_n + \tau] \vec{n} \quad (\text{III.18})$$

$$\frac{\partial}{\partial b} \vec{n} = [\Omega_n + \tau] \vec{t} - \kappa + (\text{div} \vec{n}) \vec{b} \quad (\text{III.19})$$

$$\frac{\partial}{\partial b} \vec{b} = -\theta_{bs} \vec{t} - [\kappa + (\text{div} \vec{n})] \vec{n} \quad (\text{III.20})$$

Let us now consider the main result of the paper in the form of a

**Theorem 1:** Let  $\mathcal{M}$  be a compact Riemannian 3D manifold and  $L$  the dynamo operator defined by:

$$L : \vec{u} \rightarrow (\vec{u} \cdot \nabla) \quad (\text{III.21})$$

Thus the slow dynamo condition  $p_0 = 0$  yields the Lagrangean stability given by the vanishing of the sectional curvature  $K(\vec{u}, \vec{H})$ . Fast dynamos can also be stable if the condition  $S > 0$  is fulfilled. Here the proof shall be done with the help of the following eigenvalue spectrum equation

$$L\vec{H} = p_0\vec{H} \quad (\text{III.22})$$

$\vec{H}$  obeys the self-induction equation and  $R_m \rightarrow \infty$  for a highly conducting fluid.

**Proof:** By writing the self-induction equation in the form [8]

$$\partial_t \vec{H} = R^{-1}_m \nabla^2 \vec{H} + [\vec{H}, \vec{u}] \quad (\text{III.23})$$

where  $[\vec{H}, \vec{u}]$  is the bracket defined in Riemannian 3D manifold as

$$[\vec{H}, \vec{u}] = \nabla_{\vec{H}} \vec{u} - \nabla_{\vec{u}} \vec{H} \quad (\text{III.24})$$

Taking into account the expression for the Riemann curvature in section II allow us to express the dynamo equation in terms of the Riemann curvature as

$$(\partial_t \vec{H}) \cdot \nabla \vec{u} = R(\vec{H}, \vec{u}) \vec{u} - \nabla_{\vec{H}} \nabla_{\vec{u}} \vec{u} + \nabla_{\vec{u}} \nabla_{\vec{H}} \vec{u} + R^{-1}_m \Delta \vec{H} \cdot \nabla \vec{u} \quad (\text{III.25})$$

where  $\Delta := \nabla^2$ . From these expressions one is able to compute the sectional curvature as

$$(\vec{H} \cdot \nabla) \nabla p - R^{-1}(\vec{H} \cdot \nabla) \Delta \vec{u} - R^{-1}_m (\Delta \vec{H} \cdot \nabla) \vec{u} = R(\vec{H}, \vec{u}) \vec{u} \quad (\text{III.26})$$

where we have used the Navier-Stokes viscous flow equation

$$-\nabla p + R^{-1} \Delta \vec{u} = \partial_t \vec{u} \quad (\text{III.27})$$

and  $R$  is the flow Reynolds number. Sectional curvature is then

$$K(\vec{H}, \vec{u}) = \frac{< (\vec{H} \cdot \nabla) \nabla p, \vec{H} >}{S(\vec{H}, \vec{u})} = \frac{< R(\vec{H}, \vec{u}) \vec{u}, \vec{H} >}{S(\vec{H}, \vec{u})} \quad (\text{III.28})$$

$$K(\vec{H}, \vec{u}) = \frac{\langle L\nabla p, \vec{H} \rangle}{S(\vec{H}, \vec{u})} = p_0 \frac{||\vec{H}||^2}{S} \quad (\text{III.29})$$

From this last expression the theorem is proved, since  $S$  does not vanish which implies  $p_0 = 0$ . The norm  $||\dots||$  is built from the Riemannian compact metric in the 3D manifold. Note that our computations are also in agreement with Kambe assertion that the Riemann connection would not be flat when pressure does not vanish.

Let us now consider the holonomic frame described in previous section to write the magnetic filament in the form  $\vec{H} = H_s(s, t)\vec{t}$ , which leads to the following result

$$R(\vec{H}, \vec{u}) = \langle (\vec{H} \cdot \nabla) \nabla p, \vec{H} \rangle = H_s^2 \partial_s^2 p \quad (\text{III.30})$$

where the use has been made of grad operator in the form

$$\nabla = \vec{t} \partial_s \quad (\text{III.31})$$

Thus from expression (III.31) one notes that the dynamo could be unstable if the second derivative of the filament pressure is positive. In the non-holonomic case is more complicated cause the Riemann tensor could be negative , since

$$\langle R(\vec{H}, \vec{u}) \vec{u}, \vec{H} \rangle = H_s^2 [\partial_s^2 p - \kappa(s) \partial_n p] \quad (\text{III.32})$$

where now the grad operator used in the last computation was

$$\nabla = \vec{t} \partial_s + \vec{n} \partial_n + \vec{b} \partial_b \quad (\text{III.33})$$

here in cases in which the Frenet curvature is strong as in some plasma loops in the sun [11], for example, the RHS of this equation can be negative and the Lagrangean instability of the dynamo would characterize an Anosov flow.

## IV Conclusions

In conclusion, we show that a dynamo operator defined on a compact 3D Riemannian manifold lead us to show that the slow dynamos are Lagrangean stable. It is noted that fast dynamo filaments could also be stable when the pressure of the filaments is appropriately constrained. Twisted filaments in non-holonomic frame may lead to filamentary dynamos which can be

unstable when the Frenet curvature of the plasma solar loops , for example, is too strong. A simple generalization of the work here can be made by generalizing dynamos to non-Riemannian manifolds where Cartan torsion is considered as an obstruction of the self-induction equation.

## Acknowledgements

Thanks are due to CNPq and UERJ for financial supports.

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